***Calculation Example: Boot Strapping***

**Question:** Calculate the zero coupon yields over the next 6 months, 12 months and 18 months using the government bond data below.

|  |
| --- |
| **Federal Treasury Bond Data** |
| **Maturity** | **Yield to maturity** | **Coupon rate**  | **Face value** |
| (years) | (%pa, compounding semi-annually) | (%pa, paid semi-annually) | ($) |
| 0.5 | 5 | 6 | 100 |
| 1 | 7 | 8 | 100 |
| 1.5 | 9 | 10 | 100 |
|   |   |   |   |

**Answer**: Since all of the 3 bonds’ coupons are paid semi-annually then we can assume that all yields are given as annualised percentage rates (APR’s) compounding semi-annually.

Let the coupon-paying bonds’ yields be represented by $r$. Strictly we should write $r\_{APR comp 6mths}$, but we’ll drop that notation and just write $r$.

Let the zero-coupon yields be represented by $z$. Again, strictly we should write $z\_{APR comp 6mths}$, but we’ll drop that notation. Note that we’re trying to find the spot zero coupon rates, not the forward zero coupon rates. So the spot zero coupon rate from now until 6 months in the future is $z\_{0\rightarrow 0.5yrs APR comp 6mths}$, but we’ll just write $z\_{0\rightarrow 0.5}$.

*Use the 0.5 year bond to find* $z\_{0\rightarrow 0.5}$

The 0.5 year bond pays only a single coupon at maturity, so effectively there's only one cash flow: the coupon and principal paid at maturity. Therefore its yield to maturity (YTM) is equal to the 0.5 year zero coupon spot yield per annum compounding semi-annually, which is 5% pa.

$$z\_{0\rightarrow 0.5} = r\_{0\rightarrow 0.5}= 0.05$$

While there’s no need to price this 0.5 year bond, here it is:

$$P\_{0}=\frac{C\_{0.5}+F\_{0.5}}{\left(1+{r\_{0\rightarrow 0.5}}/{2}\right)^{0.5×2}}=\frac{3+100}{\left(1+{0.05}/{2}\right)^{0.5×2}}=100.4878049$$

$$ =\frac{C\_{0.5}+F\_{0.5}}{\left(1+z\_{0\rightarrow 0.5}/2\right)^{0.5×2}}$$

*Use the 1 year bond to bootstrap and find* $z\_{0\rightarrow 1}$

The 1 year bond pays two coupons, one at 6 months and another at 1 year, together with the principal. We already know the 6 month spot zero coupon yield. We can use boot-strapping to find the 1-year spot zero-coupon yield. First we need to price the 1 year bond:

$$P\_{0, 1yr bond}=\frac{C\_{0.5}}{\left(1+r\_{0\rightarrow 1}/2\right)^{0.5×2}}+\frac{C\_{1}+F\_{1}}{\left(1+r\_{0\rightarrow 1}/2\right)^{1×2}}$$

$$ =\frac{0.08×100/2}{\left(1+0.07/2\right)^{0.5×2}}+\frac{0.08×100/2+100}{\left(1+0.07/2\right)^{1×2}}$$

 $=100.9498471$

Or use the annuity formula to present value the coupons:

$$P\_{0, 1yr bond}=\frac{C\_{0.5}}{r\_{0\rightarrow 1}/2}\left(1 -\frac{1}{\left(1+r\_{0\rightarrow 1}/2\right)^{1×2}}\right)+\frac{F\_{1}}{\left(1+r\_{0\rightarrow 1}/2\right)^{1×2}}$$

$$=\frac{0.08×100/2}{0.07/2}\left(1 -\frac{1}{\left(1+0.07/2\right)^{1×2}}\right)+\frac{100}{\left(1+0.07/2\right)^{1×2}}$$

$$=100.9498471$$

To boot-strap and solve for the 1-year spot zero-coupon yield $z\_{0\rightarrow 1}$, price the bond again but discount each coupon or principal by the appropriate zero coupon yield:

$$P\_{0, 1yr bond}=\frac{C\_{0.5}}{\left(1+z\_{0\rightarrow 0.5}/2\right)^{0.5×2}}+\frac{C\_{1}+F\_{1}}{\left(1+z\_{0\rightarrow 1}/2\right)^{1×2}}$$

$$100.9498471=\frac{0.08×100/2}{\left(1+0.05/2\right)^{0.5×2}}+\frac{0.08×100/2+100}{\left(1+z\_{0\rightarrow 1}/2\right)^{1×2}}$$

Now solve for $z\_{0\rightarrow 1}$

$$100.9498471= 3.902439024 +\frac{104}{\left(1 +z\_{0\rightarrow 1}/2\right)^{2}}$$

$$\left(1 +z\_{0\rightarrow 1}/2\right)^{2}=\frac{104}{100.9498471- 3.902439024}$$

$$z\_{0\rightarrow 1}=\left(\left(\frac{104}{100.9498471- 3.902439024}\right)^{\frac{1}{2}}-1\right)×2$$

$$ =0.070402078$$

*Use the 1.5 year bond to bootstrap and find* $z\_{0\rightarrow 1.5}$

The 1.5 year bond pays three coupons, one at 6 months, another at 1 year and the last at 1.5 years together with the principal. We already know the 6 month and 1 year spot zero coupon yields. We can use boot-strapping to find the 1.5-year spot zero-coupon yield. But first we need to price the bond using the YTM:

$$P\_{0, 1.5yr bond}=\frac{C\_{0.5}}{\left(1+r\_{0\rightarrow 1.5}/2\right)^{0.5×2}}$$

$$+\frac{C\_{1}}{\left(1+r\_{0\rightarrow 1.5}/2\right)^{1×2}}+\frac{C\_{1.5}+F\_{1.5}}{\left(1+r\_{0\rightarrow 1.5}/2\right)^{1.5×2}}$$

$$ =\frac{0.1×100/2}{\left(1+0.09/2\right)^{0.5×2}}+\frac{0.1×100/2}{\left(1+0.09/2\right)^{1×2}}+\frac{0.1×100/2+100}{\left(1+0.09/2\right)^{1.5×2}}$$

$$ =101.3744822$$

Or use the annuity formula to present value the coupons:

$$P\_{0, 1.5yr bond}=\frac{C\_{0.5}}{r\_{0\rightarrow 1.5}/2}\left(1 -\frac{1}{\left(1+r\_{0\rightarrow 1.5}/2\right)^{1.5×2}}\right)$$

$$ +\frac{F\_{1}}{\left(1+r\_{0\rightarrow 1.5}/2\right)^{1.5×2}}$$

$$=\frac{0.01×100/2}{0.09/2}\left(1 -\frac{1}{\left(1+0.09/2\right)^{1.5×2}}\right)+\frac{100}{\left(1+0.09/2\right)^{1.5×2}}$$

$$=101.3744822$$

To boot-strap and solve for the 1.5-year spot zero-coupon yield $z\_{0\rightarrow 1.5}$, price the bond again but discount each coupon or principal by the appropriate zero coupon yield:

$$P\_{0, 1.5yr bond}=\frac{C\_{0.5}}{\left(1+z\_{0\rightarrow 0.5}/2\right)^{0.5×2}}$$

$$+\frac{C\_{1}}{\left(1+z\_{0\rightarrow 1}/2\right)^{1×2}}+\frac{C\_{1.5}+F\_{1.5}}{\left(1+z\_{0\rightarrow 1.5}/2\right)^{1.5×2}}$$

$$101.3744822 =\frac{0.1×100/2}{\left(1+0.05/2\right)^{0.5×2}}$$

$$ +\frac{0.1×100/2}{\left(1+0.070402078/2\right)^{1×2}}+\frac{0.1×100/2+100}{\left(1+z\_{0\rightarrow 1.5}/2\right)^{1.5×2}}$$

Now solve for $z\_{0\rightarrow 1.5}$

$$101.374482=4.87804878+4.665740773+\frac{105}{\left(1+z\_{0\rightarrow 1.5}/2 \right)^{3}}$$

$$\left(1 +z\_{0\rightarrow 1.5}/2\right)^{3}=\frac{105}{101.374482- 4.87804878-4.665740773}$$

$$z\_{0\rightarrow 1.5}=\left(\left(\frac{105}{91.83069265}\right)^{\frac{1}{3}}-1\right)×2$$

$$ =0.091368081$$