***Risk-neutral Binomial Option Valuation***

* Main idea is that the option price now equals the expected value of the option price in the future, discounted back to the present at the risk free rate.
* Assumes investors are risk-neutral, so discount rates and expected returns are all equal to the risk free rate, the time value of money.
* Most people find it more intuitive than the no-arbitrage approach.
* Easy to apply when there’s more than one time step and multiple branches.

Last modified 16.1.17 KW

Let $S\_{t}$ be the value of the underlying stock,

$f\_{t}$ is the value of the option at time $t$,

$p$ is the probability that the stock price rises, and (1-p) is the probability that it falls,

$u$ is one plus the proportional up move in the stock price (say 1+0.1), and

$d$ is one minus the proportional down move in the stock price (say 1-0.1).

So after one period the stock price could be $S\_{1u}=S\_{0}.u$ if it went up or $S\_{1d}=S\_{0}.d$ if it went down.

***Probabilities of the Up and Down Moves***

To find the expected value of the option, we need to know the probabilities $p$ of the up and down moves in the tree branches.

This is easy in a risk-neutral world, since the stock price now must equal the present value of the expected stock price next period, discounted back by the risk-free rate.

$$S\_{0}=\left(p.S\_{tu}+\left(1-p\right)S\_{td}\right)e^{-r.t}$$

$$ =\left(p.S\_{0}.u+\left(1-p\right)S\_{0}.d\right)e^{-r.t}$$

After solving for the probability of an up move,

$p=\frac{e^{r.t}-d}{u-d}$

This probability can be used to find the value of the option:

$f\_{0}=\left(p.f\_{tu}+\left(1-p\right)f\_{td}\right)e^{-r.t}$

Notice that if the up move proportion $u$ is large, and the down move proportion $d$ is small, then the probability of an up move $p$ will be small, since the overall return on the stock must only be the risk free rate.

***Risk-neutral Valuation Steps***

1. Calculate the probability of the share price increasing. Use the pre-defined formula: $p=\frac{e^{r.t}-d}{u-d}$

Or, since it’s easier to remember, solve for $p$ using the discounted expected value of the stock:

$$S\_{0}=\left(p.S\_{tu}+\left(1-p\right).S\_{td}\right)e^{-r.t}$$



1. Calculate share prices at each node of the tree by multiplying the current share price by the up and down moves, working from left to right.



1. Calculate option values at maturity on the leaves of the tree at the very right.



1. Then find the discounted expected value of the option working from right to left, using

$$f\_{0}=\left(p.f\_{tu}+\left(1-p\right)f\_{td}\right)e^{-r.t}$$

***Choosing the Up and Down Moves***

**To Reflect Stock Price Standard Deviation**

To make the up and down moves correspond to a given stock price standard deviation $σ$ per time step,

$$u=e^{σ\sqrt{t}}$$

$$d=\frac{1}{u}=e^{-σ\sqrt{t}}$$

Where $t$ is the length of a single time step.

***Dividends***

* Dividends can easily be incorporated by making the tree branch at the times of each dividend payment. Then the dividend is subtracted from the stock price at those times.
* This is a slight modification of **step 2** in the risk-neutral valuation approach above.
* In the diagram to the right, there are two equal dividends D, at $t=1$ and $2$.
* Notice that the tree will not re-combine since the middle branches at the end are not equal.

***American Options With No Dividends***

American option prices are easily calculated by adding **step 5** to the risk-neutral valuation approach above.

**Step 5:** If the option’s intrinsic value is higher than the option’s discounted expected value (found in step 4) then the American option should be exercised. Therefore, simply choose the maximum of the option’s discounted expected value or the option’s intrinsic value.

$f\_{0 American}=max⁡(f\_{0 expected},f\_{0 intrinsic})$, where

$f\_{0 expected}=\left(pf\_{1u}+\left(1-p\right)f\_{1d}\right)e^{-rt}$, and

$f\_{0 intrinsic}=$ value of the option if exercised

For a call, $f\_{t, intrinsic,call}=(S\_{t}-K)$

***American Options With Dividends***

When valuing American options with dividends, remember that the holder will exercise just before or after the dividend to maximize their payoff.

Take care to exercise:

* Call options just **before** the dividend is paid, so the call option holder with receive the dividend.
* Put options just **after** the dividend is paid, so the put option holder will benefit from the share price falling due to the dividend, so the payoff of the put is greater.