Cost-Benefit Analysis using Net Present Value and Other Techniques

- NPV.
- IRR.
- Payback period.
- Profitability index.
- Solving for time using logarithms.

Why is this subject important?

- A quote from mathematician and physicist Albert Einstein: “Compound interest is the eighth wonder of the world. He who understands it, earns it ... he who doesn't ... pays it.”

After finishing this subject, you will be able to:

- Calculate how much you can afford to borrow to buy a house.
- Estimate the price of a house, business, share, bond or any asset.
- Discuss rental and dividend yields, capital returns, total returns, risk, hedging and inflation with confidence.
- Avoid losing money in too-good-to-be-true ventures.

- Recognise the pitfalls of applying accounting concepts naively. Accountants often overlook opportunity costs and regret sunk costs.
- Understand the effects of debt (leverage), tax and negative gearing on returns and cash flows.
- Apply mathematics to real-world financial problems.
- Help your career and find employment in finance, business, accounting, real estate, management, sales and others.
- Have more friends after struggling through an interesting and very difficult subject.

Comparing Projects

There are many ways to compare business projects including NPV, IRR, profitability index, payback period, average accounting return and others.

The best methods take into account:

- The time value of money;
- Risk; and
- The value of the project to the firm.

Net Present Value

Net Present Value (NPV) is the preferred method to value projects. It is the same as discounted cash flow (DCF) valuation.

$$NPV = V_0 = \sum_{t=0}^{T} \left(\frac{C_t}{(1+r)^t} \right)$$

$$= C_0 + \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_T}{(1+r)^T}$$

The decision criteria is that projects with positive NPV should be accepted.

Time value of money is incorporated in the discount rate.

Risk can be incorporated into NPV by increasing the discount rate (r) which generally decreases the NPV.

Positive-NPV projects add to a firm's asset value and share price.

NPV Terminology

Note that value, present value and net present value are often used interchangeably.

The value of an asset is its market value in dollars at some point in time ($ as at a certain date).

The present value (PV or V_0) is the value now (t=0), taking into account the time value of money.

Net present value (NPV) is the addition of the present values of all of the future cash flows. The NPV typically subtracts the price paid at the start from the positive cash flows received afterwards from owning the asset.

Question: A share currently trades at $10. We forecast that it:

- Will pay a $0.60 dividend in one year, after which it will be worth $11.50;
- Has a required total return of 10% pa.

Calculate the NPV of buying the share now and selling it one year later, just after the dividend is paid.

Answer: The present value of the $12.1 (=11.5+0.6) received in one year is $11 (=12.1/(1+0.1)^1).

The $10 current (t=0) market share price is already a present value.

So the NPV of buying the share would be $1 (= -10 + 11).

Calculation Example: NPV

Question: The mining firm has found a potential new gold mine on its property. The required return of the gold mine is 10% pa given as an effective annual rate. The after-tax cash flows are:

- $9m outflow to buy extra machinery needed to excavate the mine which will be delivered and paid for immediately (t=0).
- $6m inflow in one year (t=1) from gold sales.
- $5m inflow in two years (t=2) from gold sales.

Question: What is the NPV of the project and should it be accepted?

Answer:

$$NPV = V_0 = \sum_{t=0}^{T} \left(\frac{C_t}{(1+r)^t} \right)$$

$$= C_0 + \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2}$$

$$= -9m + \frac{6m}{(1+0.1)^1} + \frac{5m}{(1+0.1)^2}$$

$$= 0.58677686m = \$586{,}776.86$$

Since the NPV is positive, the project should be accepted.

Question: A mining company has $10m of assets funded by 7,500 bonds priced at $800 each and 4 million shares priced at $1 each. All figures are market values.

Nobody knows about the new gold mine discovery except a few engineers and senior management who have kept it secret. The firm is about to publically announce the details of the new gold mine. What would you expect the new share price to be?

Answer: Accepting the new mining project will increase the market value of the firm's assets by the NPV: $0.586776m.

This increase in value will not be received by the bond holders since they will only be paid the promised interest and principal payments and no more.

Equity holders have a 'residual claim' on the firm's assets. They are entitled to the extra value created by the discovery of the new gold mine. To calculate the new share price:

$$V_{old} + V_{project} \quad = \quad D \quad + \quad E$$

$$\$10m + \$0.586776m = \quad n_{bonds}.P_{bond} \quad + n_{shares}.P_{share}$$

$$\$10.586776m \quad = 0.0075m \times \$800 + 4m \times P_{share}$$

$$P_{share} = \frac{\$10.586776m - 0.0075m \times \$800}{4m}$$

$$= \$1.146694$$

If investors believe in the company's assessment of the positive NPV project, then this is the share price that we would expect to see in the market after the public announcement of the new gold mine project. This corresponds to a 14.7% capital return on the shares which were previously worth $1.

Note that a quicker way to calculate the share price increase is to just divide the project's NPV by the number of shares:

$$\Delta P_{share} = \frac{V_{project}}{n_{shares}} = \frac{\$0.586776m}{4m} = \$0.146694$$

Internal Rate of Return (IRR)

The internal rate of return is the discount rate that makes a project's NPV equal to zero.

$$NPV = C_0 + \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2}$$

$$0 = C_0 + \frac{C_1}{(1+r_{\boldsymbol{IRR}})^1} + \frac{C_2}{(1+r_{\boldsymbol{IRR}})^2}$$

The decision rule is to accept projects with an IRR (r_{IRR}) that is more than the required return of the project (r).

Project Types			
	NPV	IRR	Decision
Good, under-priced	$NPV > 0$	$IRR > r_{required}$	Accept, go ahead
Mediocre, fairly priced	$NPV = 0$	$IRR = r_{required}$	Indifferent
Bad, over-priced	$NPV < 0$	$IRR < r_{required}$	Reject, cancel

IRR is very closely related to NPV.

- If a project's IRR is greater than the required return, the NPV will be positive.
- If a project's IRR is equal to the required return, the NPV will be zero.
- If a project's IRR is less than the required return, the NPV will be negative.

Calculation Example: IRR

Question: What is the IRR of the mining project from the previous example?

Answer: The IRR is the discount rate that makes the NPV zero. Mathematically we must solve the below equation for r_{IRR}:

$$0 = C_0 + \frac{C_1}{(1 + r_{IRR})^1} + \frac{C_2}{(1 + r_{IRR})^2}$$

$$= -9m + \frac{6m}{(1 + r_{IRR})^1} + \frac{5m}{(1 + r_{IRR})^2}$$

This is a quadratic equation which we could solve algebraically to get two possible answers: 0.149829914 or -1.483163963.

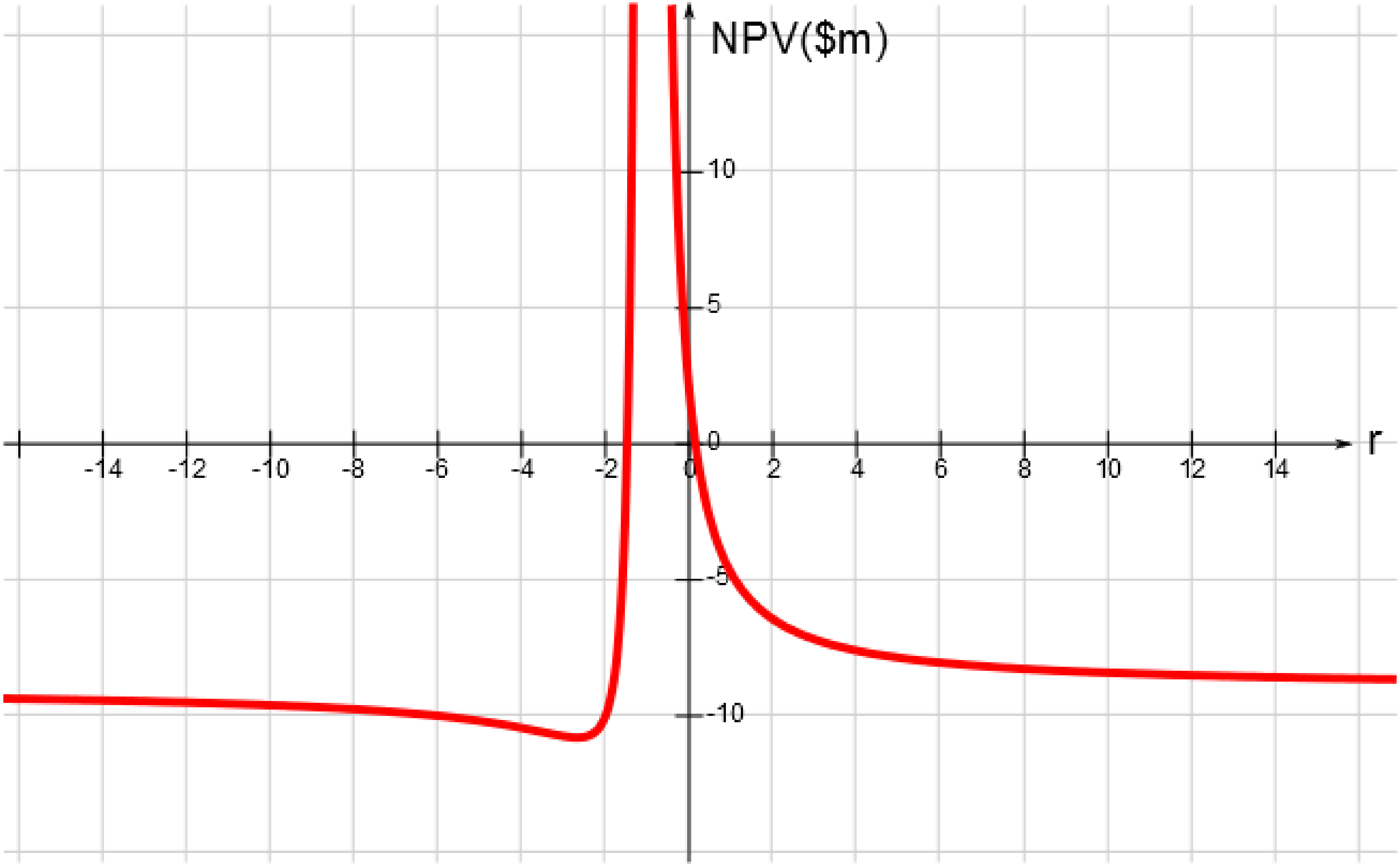
Clearly a return less than negative one (-100%) is not possible since stocks have limited liability, prices can't be negative.

The only feasible solution is that the IRR is 14.98%. Since this is more than the 10% cost of capital (same as required return), this project should be accepted.

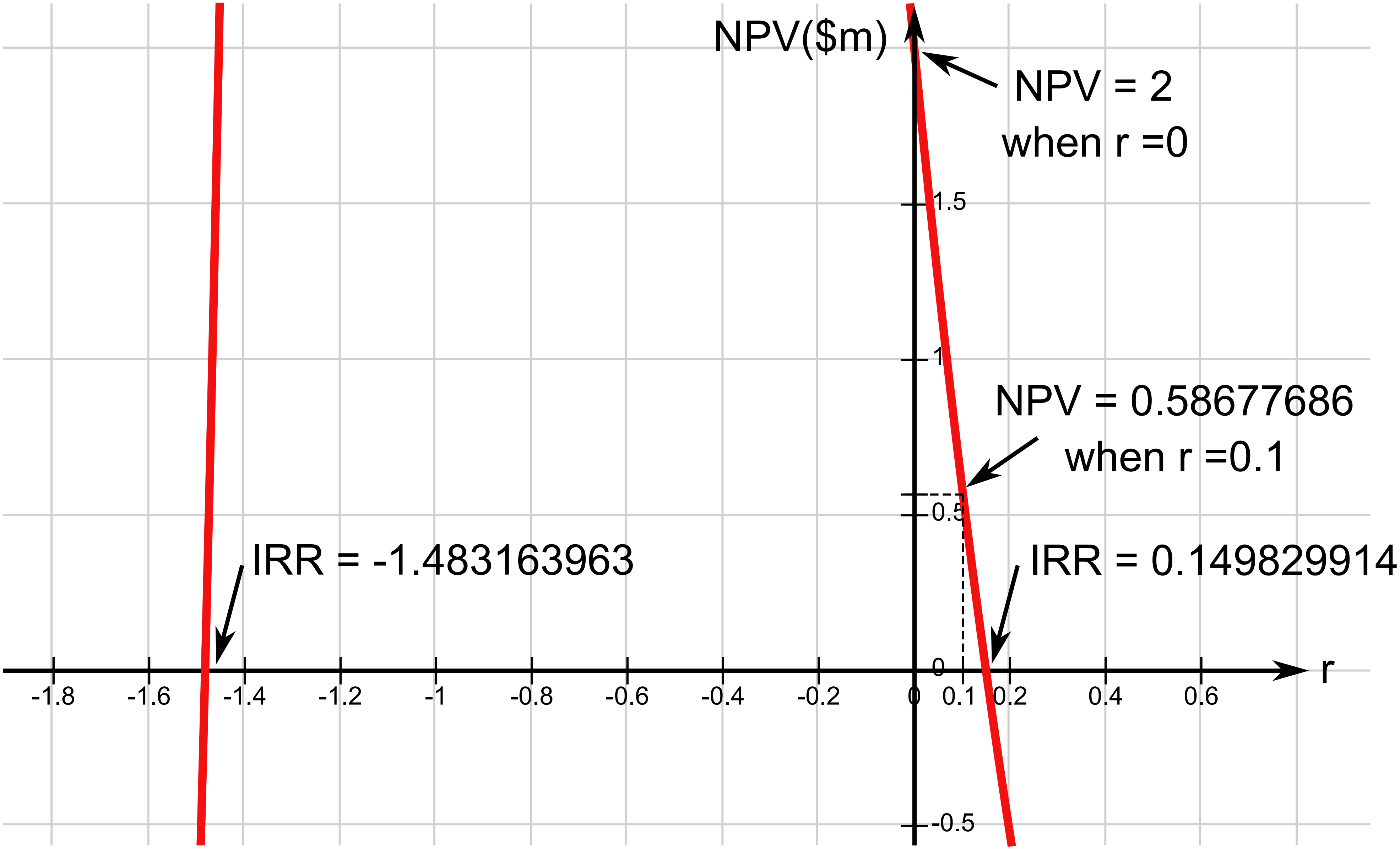
Note that it is often difficult or impossible to find the IRR algebraically if the time that the cash flows are received is any higher than 2. This is because the quadratic equation cannot be used. A mathematician would say that our problem reduces to finding the 'roots of a polynomial', which is best done with trial and error. Using a spreadsheet such as MS Excel, the function which uses trial and error automatically is '=IRR(...)'.

Here are some graphs of the problem for the visual learners.

Net Present Value (NPV) vs Discount Rate (r)



Net Present Value (NPV) vs Discount Rate (r)



IRR Problems

Similarly to NPV, IRR takes the time value of money and risk into account.

IRR is also very intuitive since people and managers are familiar with returns.

But, there are some problems with using IRR that include:

- Scale effects when comparing mutually exclusive projects.
- Multiple feasible IRR's for projects with non-conventional cash flows.

Calculation Example: IRR, Scale Effects, and Mutually Exclusive Projects

Question: A developer owns a block of land next to a highway. He can:

- Build a restaurant at a cost of $1m now which will earn $0.2m paid at the end of every year forever; Or, he can
- Build an apartment block at a cost of $10m now which will earn $1.5m paid at the end of every year forever.

He cannot build both, the local government won't allow it.

Both projects have the same level of risk and therefore the same cost of capital which is 10% pa. Which project should the developer pick?

Answer using IRR:

To calculate the IRR of the restaurant:

$$V_0 = C_0 + \frac{C_{1,2,3,\ldots}}{r}$$

$$0 = -1m + \frac{0.2m}{r_{IRR}}$$

$$r_{IRR} = \frac{0.2m}{1m} = 0.2 = 20\%$$

To calculate the IRR of the apartments:

$$V_0 = C_0 + \frac{C_{1,2,3,\ldots}}{r}$$

$$0 = -10m + \frac{1.5m}{r_{IRR}}$$

$$r_{IRR} = \frac{1.5m}{10m} = 0.15 = 15\%$$

Since the restaurant has the higher IRR, it looks like a better idea than the apartments. But this is a bad conclusion! Let's find the NPV's to see why.

Answer using NPV:

To calculate the NPV of the restaurant:

$$V_0 = C_0 + \frac{C_{1,2,3,\ldots}}{r}$$

$$= -1m + \frac{0.2m}{0.1} = 1m$$

To calculate the NPV of the apartments:

$$V_0 = C_0 + \frac{C_{1,2,3,\ldots}}{r}$$

$$= -10m + \frac{1.5m}{0.1} = 5m$$

Land Development Project Details		
	Restaurant	**Apartment block**
Initial investment ($m)	1	10
Perpetual annual cash flow ($m)	0.2	1.5
NPV ($m)	1	5
IRR (pa)	20%	15%

The apartments have a higher NPV than the restaurant, so the apartments will create more wealth for the developer, even though they have a lower internal rate of return (IRR).

The problem with using the IRR technique here is that the projects are mutually exclusive. The apartments are much bigger than the restaurant, so the IRR method leads to an incorrect conclusion about which project is better.

Calculation Example: Non-conventional Cash Flows and Multiple Feasible IRR's

Question: The mining firm has found ***another*** potential new gold mine on its property. The required return of the gold mine is 10% pa given as an effective annual rate. The after-tax cash flows are:

- $9m outflow to buy extra machinery needed to excavate the mine which will be delivered and paid for immediately (t=0).
- $13.9m inflow in one year (t=1) from gold sales.
- $10m inflow in two years (t=2) from gold sales.

- $15m outflow in two years (t=3) to clean up the mine and restore the natural environment.

Evaluate the project using the NPV and IRR methods.

Notice that there is a negative cash flow at the end of the project (t=3). This is a common type of non-conventional cash flow.

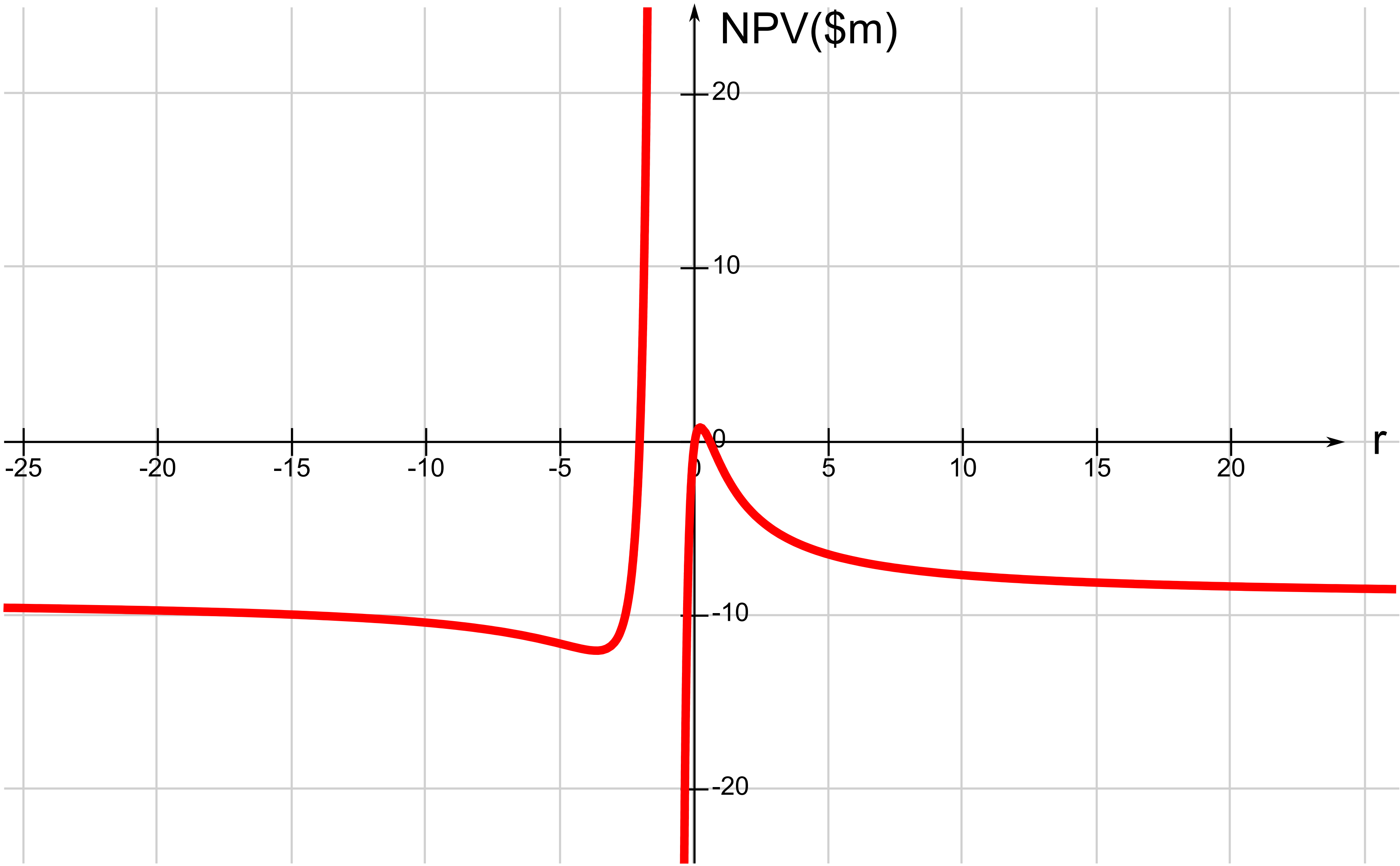
Answer: In this particular case there are actually 3 internal rates of returns! You can see them in the graph. The left-most IRR is unfeasible since it's less than -1. But the other two, 0.937% and 58.009% are perfectly feasible. So which one is the right one to compare to the 10% cost of capital?

Since the NPV is positive between 0.937% and 58.009%, the project should be accepted for any cost of capital between those rates. Therefore we should accept the project.

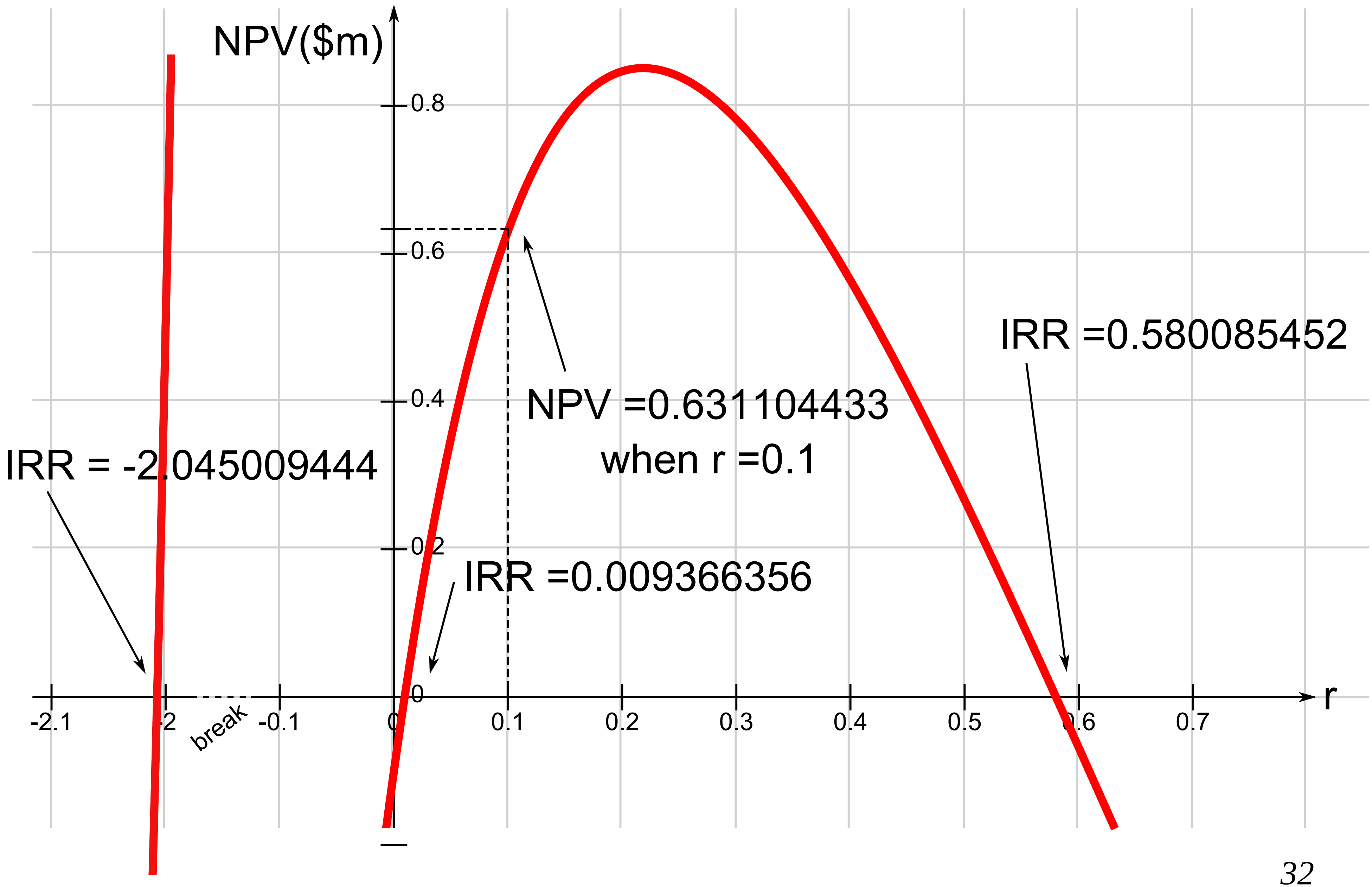
You can see that if we naively evaluated the IRR using a spreadsheet program's IRR function we may have been given a value of 0.937% and then rejected the project since it is less than the 10% required return. Of course, this would be the wrong thing to do.

In case you're interested, this is how the NPV vs discount rate graph looks like from a zoomed-out perspective and a close-up perspective.

Net Present Value (NPV) vs Discount Rate (r)



Net Present Value (NPV) vs Discount Rate (r)



Pay-back Period

Payback period is measured in years and shows how long the project takes to 'pay itself off'. In other words, how many years it is expected to take to re-coup the cost of the project and break even.

Projects with shorter payback periods are preferred. Sometimes managers use a decision rule that any project with a payback period above a threshold number of years should be rejected.

Pay-back Period: Pros and Cons

The advantage of the payback period approach is that it is intuitive, simple to understand and simple to calculate.

The disadvantages are that it:

- Doesn't explicitly take the time value of money or risk into account.
- Provides no indication about how much more the firm will be worth if the project is accepted.
- Ignores all cash flows after the payback period.
- Suffers from the same scale effect problems as IRR when ranking mutually exclusive projects.

Calculation Example: Pay-back Period

Question: A mining firm's potential new gold mine has the following after-tax cash flows:

- -$9m to buy extra machinery needed to excavate the mine which will be delivered and paid for immediately (t=0).
- $6m in one year (t=1) from gold sales.
- $5m in two years (t=2) from gold sales.

Question: What is the payback period, assuming that the cash flows are received (or paid) in full at the given time?

Answer: The \$9m cost will be paid back at t=2 since the cumulative cash flow at t=2 will be positive (>0). Note that present values are not calculated, we just sum up the cash flows as if we're accountants:

$$
\begin{aligned}
C_{cumulative,t=0\to 2} &= C_0 + C_1 + C_2 \\
&= -9m + 6m + 5m \\
&= 2m \\
&> 0
\end{aligned}
$$

Question: What is the payback period, assuming that all cash flows are received smoothly over the year before the given time (but assume that the negative cash flow at the start is paid in full at t=0)?

So the $6m at time 1 is actually earned smoothly from t=0 to t=1.

Answer: Make a new column that sums the current and past cash flows at each time, called ‘Cumulative cash flows’:

Payback Period Calculation		
Time (yrs)	Cash flow ($m)	Cumulative cash flow ($m)
0	-9	-9
1	6	-3
2	5	2

The payback period clearly occurs sometime during the second year (between t=1 and 2).

The payback period is the time at which the first positive cumulative cash flow occurs, less the positive cumulative cash flow divided by the single cash flow in that period:

$$\mathrm{T}_{\text{pay back}} = \begin{pmatrix} Time\ of \\ first\ positive \\ cumulative \\ cash\ flow \end{pmatrix} - \frac{\begin{pmatrix} first\ positive \\ cumulative \\ cash\ flow \end{pmatrix}}{\begin{pmatrix} cash\ flow \\ in\ that\ year \end{pmatrix}}$$

$$= 2 - \frac{(-9 + 6 + 5)}{5}$$

$$= 2 - \frac{2}{5}$$

$$= 1.6yrs$$

Profitability Index

Profitability index is calculated as:

$$\text{PI} = \frac{\text{NPV(future cash flows excluding the initial investment)}}{\text{Initial investment at time zero}}$$

Projects are accepted if their profitability index is more than one. The bigger the profitability index the better.

The profitability index is simple to understand, but since it's a proportional measure, not a dollar value measure, it suffers from the same scale effect problem as the internal rate of return (IRR) method.

Calculation Example: Profitability Index

Question: A mining firm's potential new gold mine has the following after-tax cash flows:

- \$9m outflow to buy extra machinery needed to excavate the mine which will be delivered and paid for immediately (t=0).
- \$6m inflow in one year (t=1) from gold sales.
- \$5m inflow in two years (t=2) from gold sales.

The discount rate is 10% pa given as an effective annual rate.

What is the profitability index?

Answer: Remember that an investment is a cash outflow, just the same as a cost. So a positive investment is a negative cash flow:

$$Initial\ investment = -(C_0) = -(-9m) = \$9m$$

NPV(future cash flows excluding the initial investment)

$$= \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2}$$

$$= \frac{6m}{(1+0.1)^1} + \frac{5m}{(1+0.1)^2}$$

$$= \$9.58677686m$$

$$\mathrm{PI} = \frac{\text{NPV(future cash flows excluding the initial investment)}}{\text{Initial investment at time zero}}$$

$$= \frac{\$9.58677686m}{\$9m} = 1.065197429$$

Since the profitability index is more than one, the project should be accepted.

Questions: Profitability index

http://www.fightfinance.com/?q=45,174,191,219,

Calculation Example: Solving for Time - Logarithms

Question: You have some money in the bank. The effective monthly interest rate is 0.5% per month. How long will it take before your money in the bank has doubled?

Answer: Let V_0 be the money currently in the bank. Therefore the money in the bank will double when the future value V_t equals $2V_0$. The effective monthly rate is 0.5% which is 0.005.

$$V_t = V_0(1 + r)^t$$

$$2V_0 = V_0(1 + 0.005)^t$$

$$2 = (1 + 0.005)^t$$

$$1.005^t = 2$$

$$\log(1.005^t) = \log(2)$$

$$t \times \log(1.005) = \log(2)$$

$$t = \frac{\log(2)}{\log(1.005)} = 138.98 \text{ months} = 11.58 \text{ years.}$$

Question: You want to borrow \$500,000 now to buy a house. You can afford to pay \$4,000 per month towards the mortgage. The interest rate on the mortgage is 9% pa, given as an annualised percentage rate compounding per month. Therefore the effective monthly rate is 0.75% per month (=0.09/12). How long will it take you to pay it off?

Answer: The mortgage is fully amortising since the payments must completely pay off the loan at maturity. So,

$$V_0 = \frac{C_1}{r}\left(1 - \frac{1}{(1+r)^T}\right)$$

$$500{,}000 = \frac{4{,}000}{(0.09/12)}\left(1 - \frac{1}{(1+0.09/12)^T}\right)$$

$$\frac{500{,}000}{4{,}000} = \frac{1}{(0.0075)}\left(1 - \frac{1}{(1 + 0.0075)^T}\right)$$

$$\frac{500 \times 0.0075}{4} = 1 - \frac{1}{1.0075^T}$$

$$0.9375 = 1 - \frac{1}{1.0075^T}$$

$$0.0625 = \frac{1}{1.0075^T}$$

$$1.0075^T = \frac{1}{0.0625}$$

$$= 16$$

$\log(1.0075^T) = \log(16)$, and using log rules,

$t \times \log(1.0075) = \log(16)$

$$t = \frac{\log(16)}{\log(1.0075)} = 371.06 \text{ months} = 30.92 \text{ years}$$