Calculation Example: Solving for Time - Logarithms

Question: You have some money in the bank. The effective monthly interest rate is 0.5% per month. How long will it take before your money in the bank has doubled?

Answer: Let V_0 be the money currently in the bank. Therefore the money in the bank will double when the future value V_t equals $2V_0$. The effective monthly rate is 0.5% which is 0.005.

$$V_t = V_0(1 + r)^t$$

$$2V_0 = V_0(1 + 0.005)^t$$

$$2 = (1 + 0.005)^t$$

$$1.005^t = 2$$

$$\log(1.005^t) = \log(2)$$

$$t \times \log(1.005) = \log(2)$$

$$t = \frac{\log(2)}{\log(1.005)} = 138.98 \text{ months} = 11.58 \text{ years.}$$

Question: You want to borrow \$500,000 now to buy a house. You can afford to pay \$4,000 per month towards the mortgage. The interest rate on the mortgage is 9% pa, given as an annualised percentage rate compounding per month. Therefore the effective monthly rate is 0.75% per month (=0.09/12). How long will it take you to pay it off?

Answer: The mortgage is fully amortising since the payments must completely pay off the loan at maturity. So,

$$V_0 = \frac{C_1}{r}\left(1 - \frac{1}{(1+r)^T}\right)$$

$$500{,}000 = \frac{4{,}000}{(0.09/12)}\left(1 - \frac{1}{(1+0.09/12)^T}\right)$$

$$\frac{500{,}000}{4{,}000} = \frac{1}{(0.0075)}\left(1 - \frac{1}{(1+0.0075)^T}\right)$$

$$\frac{500 \times 0.0075}{4} = 1 - \frac{1}{1.0075^T}$$

$$0.9375 = 1 - \frac{1}{1.0075^T}$$

$$0.0625 = \frac{1}{1.0075^T}$$

$$1.0075^T = \frac{1}{0.0625}$$

$$= 16$$

$\log(1.0075^T) = \log(16),\;$ and using log rules,

$$\mathrm{t} \times \log(1.0075) = \log(16)$$

$$t = \frac{\log(16)}{\log(1.0075)} = 371.06 \text{ months} = 30.92 \text{ years}$$