## Sum and Product functions

The capital Greek letter sigma denotes the sum of T discrete values (from $\mathrm{t}=1$ to T ) of some variable x :
$\sum_{t=1}^{T} x=x_{1}+x_{2}+\cdots+x_{T}$

The capital Greek letter pi denotes the product of T discrete values (from $\mathrm{t}=1$ to T ) of some variable x :
$\prod_{t=1}^{T} x=x_{1} \times x_{2} \times \ldots \times x_{T}$

## Single Period Returns

Gross discrete return, often denoted by capital R:
$G D R_{0 \rightarrow 1}=\frac{p_{1}}{p_{0}}$
Net discrete return, also called the effective return $\left(r_{e f f}\right)$ or relative return, often denoted by lower case r:
$N D R_{0 \rightarrow 1}=\frac{p_{1}}{p_{0}}-1=\frac{p_{1}-p_{0}}{p_{0}}=G D R_{0 \rightarrow 1}-1$
Log gross discrete return, also called the continuously compounded return ( $r_{c c}$ ) or force of interest (y):
$L G D R_{0 \rightarrow 1}=\ln \left(\frac{p_{1}}{p_{0}}\right)=\ln \left(G D R_{0 \rightarrow 1}\right)$

## Geometric and Arithmetic Averages

Arithmetic average over T periods:
$\bar{x}_{\text {arithmetic average }, 0 \rightarrow T}=\frac{\sum_{t=1}^{T}\left(x_{t-1 \rightarrow t}\right)}{T}$
$=\frac{x_{0 \rightarrow 1}+x_{1 \rightarrow 2}+x_{2 \rightarrow 3}+\cdots+x_{T-1 \rightarrow T}}{T}$
Geometric average over T periods:
$\bar{x}_{\text {geometric average }, 0 \rightarrow T}=\left(\prod_{t=1}^{T}\left(x_{t-1 \rightarrow t}\right)\right)^{\frac{1}{T}}$
$=\left(x_{0 \rightarrow 1} \cdot x_{1 \rightarrow 2} \cdot x_{2 \rightarrow 3} \ldots x_{T-1 \rightarrow T}\right)^{\frac{1}{T}}$

## Geometric and Arithmetic Average Returns

Two types of average (also called mean):

Arithmetic average net discrete return (NDR) from time 0 to $n$ :
$\bar{r}_{\text {arithm NDR } 0-n}=\frac{\sum_{i=1}^{n}\left(r_{i}\right)}{n}=\frac{r_{1}+r_{2}+\cdots+r_{n}}{n}$

Geometric average net discrete return (NDR) from time 0 to $n$ :

$$
\begin{gathered}
\bar{r}_{\text {geom NDR } 0-n}=\left[\left(1+r_{1}\right)\left(1+r_{2}\right) \ldots\left(1+r_{n}\right)\right]^{1 / n}-1 \\
=\left(\frac{p_{n}}{p_{0}}\right)^{1 / n}-1
\end{gathered}
$$

## Average Returns - A Curious Example

| Time (year) | $\mathbf{0}$ | $\mathbf{1}$ | $\mathbf{2}$ |
| :--- | :---: | :---: | :---: |
| Share price (\$) | 100 | 50 | 100 |
| Net discrete return pa |  | -0.5 | 1.0 |

Note that we will express all returns as pure decimals, not $\%$, unless marked as so.
Similarly for standard deviation and variance.


$$
\begin{aligned}
\bar{r}_{\text {geom NDR 0-2,p.a. }} & =\left[\left(1+r_{1}\right)\left(1+r_{2}\right) \ldots\left(1+r_{n}\right)\right]^{1 / n}-1 \\
& =\left[\left(1+r_{0-1}\right)\left(1+r_{1-2}\right)\right]^{1 / 2}-1 \\
& =[(1+-0.5)(1+1)]^{1 / 2}-1 \\
& =0 \\
\bar{r}_{\text {arithm NDR } 0-2, \text { p.a. }} & =\frac{r_{1}+r_{2}+\cdots+r_{n}}{n} \\
& =\frac{r_{0-1}+r_{1-2}}{2} \\
& =\frac{-0.5+1}{2} \\
& =0.25
\end{aligned}
$$

Which average return tells the true story?

If an investor wants to buy the share at time zero and sell it two years later, then the geometric average return makes more sense since it takes compounding into account over time. If the investor wants to buy for one year and sell a year later, so there's a $1 / 2$ probability of losing $50 \%$ and a $1 / 2$ probability of gaining $100 \%$ then the arithmetic average is arguably better.

In finance we frequently use both types of averages.

- Arithmetic average returns are often used in Markowitz mean-variance portfolio analysis to find stocks' average returns from past data.
- Geometric averages are often used in the debt markets for computing the term structure of interest rates. Notice that the quantity $\left(1+r_{0 \rightarrow 1 \text { eff }}\right)$ is the gross discrete return (GDR).


## Term Structure of Interest Rates: The Expectations Hypothesis

Expectations hypothesis is that long term spot rates (plus one) are the geometric average of the shorter term spot and forward rates (plus one) over the same time period.
Mathematically:

$$
1+r_{0 \rightarrow T}=\left(\left(1+r_{0 \rightarrow 1}\right)\left(1+r_{1 \rightarrow 2}\right)\left(1+r_{2 \rightarrow 3}\right) \ldots\left(1+r_{(T-1) \rightarrow T}\right)\right)^{\frac{1}{T}}
$$

or
$\left(1+r_{0 \rightarrow T}\right)^{T}=\left(1+r_{0 \rightarrow 1}\right)\left(1+r_{1 \rightarrow 2}\right)\left(1+r_{2 \rightarrow 3}\right) \ldots\left(1+r_{(T-1) \rightarrow T}\right)$
Where T is the number of periods and all rates are effective rates over each period.

