## Term Structure of Interest Rates

Long term interest rates are based on expectations of future short term interest rates.

We will discuss spot and forward rates, yield curves, and then two important theories of interest rates:

- Expectations hypothesis
- Liquidity premium theory

Another theory that we won't discuss is the 'segmented markets' theory.

## Spot and Forward Interest Rates

Spot rate: An interest rate measured from now until a future time. For example, a 3-year zero-coupon bond with a yield of $8 \%$ pa has a 3 -year pa spot rate of $r_{0-3 \text {,yearly }}=0.08$ pa. Note that spot rates can be from now until any future time.

Forward rate: An interest rate measured from a future time until a more distant future time. For example, if a company promised, one year from now, to issue a 3-year zero-coupon bond with a yield of $8 \%$ pa, then the forward rate from years 1 to 4 would be $r_{1-4, \text { yearly }}=0.08$. Forward rates are sometimes written with an ' $f$ ' rather than 'r'.

Spot and forward rates can be quoted as APR's or effective rates.

## Term Structure of Interest Rates: The Expectations Hypothesis

Expectations hypothesis is that long term spot rates (plus one) are the geometric average of the shorter term spot and forward rates (plus one) over the same time period.
Mathematically:
$1+r_{0 \rightarrow T}=\left(\left(1+r_{0 \rightarrow 1}\right)\left(1+r_{1 \rightarrow 2}\right)\left(1+r_{2 \rightarrow 3}\right) \ldots\left(1+r_{(T-1) \rightarrow T}\right)\right)^{\frac{1}{T}}$
or
$\left(1+r_{0 \rightarrow T}\right)^{T}=\left(1+r_{0 \rightarrow 1}\right)\left(1+r_{1 \rightarrow 2}\right)\left(1+r_{2 \rightarrow 3}\right) \ldots\left(1+r_{(T-1) \rightarrow T}\right)$
Where T is the number of periods and all rates are effective rates over each period.

## Calculation Example: Term Structure of Interest Rates

Question: The following US Government Bond yields were quoted on $5 / 3 / 2012$ (sourced from Bloomberg):

- 6 -month zero-coupon bonds yielded $0.11 \%$.
- 12 -month zero-coupon bonds yielded $0.16 \%$.

Find the forward rate from month 6 to 12. Quote your answer as a yield in the same form as the above yields are quoted.

Remember that US (and Australian) bonds normally pay semiannual coupons.

Answer: Even though these are zero-coupon bonds, since they are US bonds the yield would be quoted as an APR compounding semi-annually since all coupon bonds pay semiannual coupons. This means that our answer should be quoted in the same form, as an APR compounding semi-annually.

Therefore we have to convert the APR compounding every 6 months to an effective 6 month yield by dividing it by 2 .
$r_{0 \rightarrow 0.5 y r, e f f} 6 \mathrm{mth}=\frac{r_{0 \rightarrow 0.5 y r, A P R} \text { comp semi-annually }}{2}$

$$
=\frac{0.0011}{2}=0.00055
$$

$$
\begin{aligned}
r_{0 \rightarrow 1 y r, \text { eff } 6 \mathrm{mth}} & =\frac{r_{0 \rightarrow 1 y r, A P R \text { comp semi-annually }}}{2} \\
& =\frac{0.0016}{2}=0.0008
\end{aligned}
$$

We want to find $r_{0.5 y r \rightarrow 1 y r, e f f} 6 m t h$, which is the effective 6 month forward rate over the second 6 month period ( 0.5 years to 1 year).
Applying the term structure of interest rates equation:

$$
\begin{aligned}
& \left(1+r_{0 \rightarrow T}\right)^{T}=\left(1+r_{0 \rightarrow 1}\right)\left(1+r_{1 \rightarrow 2}\right)\left(1+r_{2 \rightarrow 3}\right) \ldots\left(1+r_{(T-1) \rightarrow T}\right) \\
& \left(1+r_{0 \rightarrow 1 \text { yr,eff } 6 m t h}\right)^{2}=\left(1+r_{0 \rightarrow 0.5 y r, e f f} 6 m t h\right)\left(1+r_{0.5 \rightarrow 1 \text { yr,eff } 6 m t h}\right) \\
& (1+0.0008)^{2}=(1+0.00055)\left(1+r_{0.5 \rightarrow 1 y r, e f f} 6 m t h\right)
\end{aligned}
$$

$$
\begin{aligned}
r_{0.5 \rightarrow 1 y r, \text { eff } 6 \mathrm{mth}} & =\frac{(1+0.0008)^{2}}{(1+0.00055)}-1 \\
& =0.001050062
\end{aligned}
$$

But this is an effective 6 month rate. Let's convert it to an APR compounding every 6 months.
$r_{0.5 \rightarrow 1 y r, A P R ~ c o m p ~}^{6 m t h s}=r_{0.5 \rightarrow 1 y r, \text { eff } 6 \mathrm{mth}} \times 2$

$$
\begin{aligned}
& =0.001050062 \times 2 \\
& =0.002100124=0.21 \% p a
\end{aligned}
$$

Note that this forward rate APR from 0.5 years to 1 year is bigger than both of the bond yield APR's (which are spot rates). This makes sense since we have a normal upward sloping yield curve ( $r_{0 \rightarrow 0.5 y r}<r_{0 \rightarrow 1 y r}$ ) so the forward rate
$\left(r_{0.5 \rightarrow 1 y r}\right)$ should be greater than the spot rates
$\left(r_{0 \rightarrow 0.5 y r}\right.$ and $\left.r_{0 \rightarrow 1 y r}\right)$.

## Quick method: Convert rates in the term structure equation

 Interest rate conversion from anualised percentage rates (APR's) to effective returns can be a headache. Most people prefer to do the APR to effective rate conversion in the expectations formula itself:$$
\begin{aligned}
& \left(1+r_{0 \rightarrow T \text { eff }}\right)^{T}=\left(1+r_{0 \rightarrow 1 \text { eff }}\right)\left(1+r_{1 \rightarrow 2 \text { eff }}\right) \ldots\left(1+r_{(T-1) \rightarrow T \text { eff }}\right) \\
& \left(1+r_{0 \rightarrow 1 y r, \text { eff } 6 m t h}\right)^{2}=\left(1+r_{0 \rightarrow 0.5 y r, \text { eff } 6 m t h}\right)\left(1+r_{0.5 \rightarrow 1 y r, \text { eff } 6 m t h}\right) \\
& \left(1+\frac{r_{0 \rightarrow 1 y r, A P R ~} 6 m t h s}{2}\right)^{2}=\left(1+\frac{r_{0 \rightarrow 0.5 y r, A P R ~ 6 m t h s}}{2}\right)\left(1+\frac{r_{0.5 \rightarrow 1 y r, A P R ~} \text { mths }}{2}\right) \\
& \left(1+\frac{0.0016}{2}\right)^{2}=\left(1+\frac{0.0011}{2}\right)\left(1+\frac{r_{0.5 \rightarrow 1 y r, A P R 6 m t h s}}{2}\right)
\end{aligned}
$$

$r_{0.5 \rightarrow 1 y r, A P R ~ 6 m t h s}=\left(\frac{\left(1+\frac{0.0016}{2}\right)^{2}}{\left(1+\frac{0.0011}{2}\right)}-1\right) \times 2=0.002100124$
This forward rate from 6 months to one year is $0.21 \%$ pa given as an APR compounding every 6 months.

## Yield Curves

Yield curves show the behaviour of short and long-term interest rates and can give an indication of expected future interest rates.

Yield curves can be flat, normal, inverse, or some combination. The x-axis of a yield curve is the time to maturity of the bond, and the $y$-axis is the yield of the bond.

A flat yield curve is a straight horizontal line. Short and long term spot rates are equal, and yearly spot and forward rates are also equal. Other names for flat interest rates are 'constant', 'unchanging', or 'level' interest rates.

A normal yield curve is an upward sloping line or curve. Short term spot rates are less than long term spot rates. Yearly spot rates are less than yearly forward rates. Other names for normal yield-curves are 'upward sloping' or 'steep ' yield curves. These yield curves are 'normal' since yields usually exhibit this behaviour.

An inverse yield curve is a downward sloping line or curve. Short term spot rates are more than long term spot rates. Yearly spot rates are more than yearly forward rates. Other names for inverse yield-curves are 'downward sloping' or 'inverted ' yield curves.


## An Extension: Liquidity Premium Theory

The expectations hypothesis assumes that investors are indifferent between investing in a 10 year bond, or investing in a one year bond, then investing in another 1 year bond after the first is repaid, and so on for 10 years.

Most investors would prefer to lend lots of short term bonds repeatedly rather than one big long one. The reason is that the long-term bond locks up the investor's cash and she loses the option to change her mind and do something else with the cash.

The liquidity premium theory suggests investors are only enticed to lend their cash out long-term if they are rewarded
for doing so in the form of higher long-term rates compared to short term rates. This means that forward rates will be higher than the expected spot rates over the same time period.

For example, if the forward rate from years 1 to 2 is $8 \%$ now, then 1 year later the spot rate (from years 0 to 1 ) would tend to be less, say $7.5 \%$.

This theory explains why the up-ward sloping yield curve is normal, since spot rates would tend to be less than forward rates.

## Real World Example: Yield Curves and Term Structure of Interest Rates

See the below sources for an interesting view of yield curves and the term structure of interest rates.

Australian Federal Government (Fixed Coupon) Bond Yields:
http://www.bloomberg.com/markets/rates-
bonds/government-bonds/australia

## Table of yields on evening of $5 / 3 / 2012$. Source: Bloomberg.

| Australian Government Bonds |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | COUPON | MATURITY | PRICE/YIELD | PRICE/YIELD CHANGE | TIME |
| 3-Month | 0.000 | 06/08/2012 | 4.15 / 4.15 | 98.943 / 4.150 | 02/24 |
| 1-Year | 6.500 | 05/15/2013 | 103.06 / 3.83 | 0.058 / -0.055 | 00:39 |
| 2-Year | 6.250 | 06/15/2014 | 105.48 / 3.71 | 0.131/-0.061 | 00:39 |
| 3-Year | 6.250 | 04/15/2015 | 107.39 / 3.71 | $0.181 /-0.062$ | 00:39 |
| 4-Year | 4.750 | 06/15/2016 | 103.87 / 3.76 | 0.237 / -0.060 | 00:39 |
| 5-Year | 6.000 | 02/15/2017 | 109.95 / 3.77 | 0.284 / -0.061 | 00:39 |
| 6-Year | 5.500 | 01/21/2018 | 108.60 / 3.85 | 0.374 / -0.069 | 00:39 |
| 7-Year | 5.250 | 03/15/2019 | 108.25 / 3.90 | 0.452 / -0.071 | 00:39 |
| 8-Year | 4.500 | 04/15/2020 | 103.63 / 3.97 | $0.537 /-0.077$ | 00:39 |
| 10-Year | 5.750 | 05/15/2021 | 113.05 / 4.04 | 0.649 / -0.080 | 00:39 |
| 15-Year | 4.750 | 04/21/2027 | 103.53 / 4.43 | 0.931 / -0.084 | 00:38 |

Orange line: current yield, Green line: previous close (yesterday's) yield. As at 5/3/2012. Note the humped curve.


