***The Risk Free Rate (rf) and the Minimum Variance Set of All Assets***

* Risk-free securities have zero standard deviation of returns. Government bonds (also called Treasuries) are assumed to be risk free securities.
* The return of the risk-free security is referred to as $r\_{f}$. It’s also used to refer to the security itself.
* When $r\_{f}$ is included, the **new MVS becomes a line** from $r\_{f}$ through the tangency portfolio (T) on the Markowitz bullet.
* $r\_{f}$ is a constant, so it has zero variance, and zero covariance with other securities.
* $r\_{f}=E\left(r\_{f}\right)=μ\_{rf}$, since $r\_{f}$ is a constant.

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***The Tangency Portfolio***

* The tangency portfolio T is the only risky portfolio worth investing in. It is comprised of the stocks A, B and C.
* For any level of risk (standard deviation), the highest return possible can be achieved by investing in T and $r\_{f}$.
* Lines from $r\_{f}$ through any portfolio are called Capital Allocation Lines.
* The CAL through T has the best risk-return trade off. It has the steepest gradient (rise/run).
* The gradient of the CAL is also called the Sharpe ratio.

***Sharpe Ratio***

The **Sharpe ratio** (S) of a stock is the gradient of the line from $r\_{f}$ through the stock. It is the gradient of the stock’s CAL.

$S\_{i}=\frac{μ\_{i}-r\_{f}}{σ\_{i}}$

Where $S\_{i}$ is the Sharpe ratio of stock ‘i’, $μ\_{i}$ is its expected return and $σ\_{i}$ is its standard deviation.

In the diagram, portfolio T’s Sharpe ratio is greater than stock B’s since the CAL through T is steeper. Therefore portfolio T is preferable to stock B.

***The Market Portfolio***

* The market portfolio M is the tangency portfolio of **all** risky assets.



* The line through M and $r\_{f}$is called the Capital Market Line (CML).
* The CML has the steepest gradient, therefore the market portfolio has the highest Sharpe ratio.

***Equation of the Capital Market Line (CML)***

**$μ=r\_{f}+σ\left(\frac{μ\_{m}-r\_{f}}{σ\_{m}}\right)$

This is easy to see since the:

* y-axis is expected return $μ$
* x-axis is standard deviation $σ$
* y-intercept is $r\_{f}$
* gradient $\frac{rise}{run}$ between $r\_{f}$ and M is $\frac{μ\_{m}-r\_{f}}{σ\_{m}}$

So in $y=mx+b$ form:

$$μ=\left(\frac{μ\_{m}-r\_{f}}{σ\_{m}}\right)σ+r\_{f}$$

***Calculation Example***

**Question**: Assume a 3-stock world consisting of A, B and C, as well as the risk free security. The market portfolio has been calculated to have weights 1/3 in each of A, B and C. The risk free rate $r\_{f}=0.05$, the market return is $r\_{m}=0.3$ and the market’s standard deviation is $σ\_{m}=0.2$.

Find the weights in stocks A, B, C and $r\_{f}$ which makes an efficient portfolio (P) with a return of $r\_{p target}=0.1$. Also find this portfolio’s standard deviation $σ\_{p target}$.

**Answer**: An efficient portfolio has minimum variance (or st. dev.) for a given return. All portfolios on the CML are efficient. Therefore we need only consider investing in the market portfolio (M) and the risk free rate ($r\_{f}$).

To find the weights we need to invest in M and $r\_{f}$, we will use the portfolio return equation, with a target portfolio return of 0.1:

$$r\_{P}=x\_{1}.r\_{1}+x\_{2}.r\_{2}+…+x\_{n}.r\_{n}$$

$$0.1=x\_{M}×0.3 + x\_{rf}×0.05$$

Now we’re stuck since we have 2 unknowns ($x\_{M}$ and $x\_{rf}$) and only one equation so we can’t find either of the weights. But there is another equation, the ‘sum of the weights equals one’:

$$x\_{1}+x\_{2}+…+x\_{n}=1$$

$$x\_{M}+x\_{rf}=1$$

$$x\_{rf}=1-x\_{M}$$

Substitute this into the portfolio return equation to get:

$0.1=x\_{M}×0.3 +(1-x\_{M})×0.05$

$$x\_{M}=0.2$$

So, $x\_{rf}=1-0.2=0.8$

This makes sense since the target return of 0.1 is closer to $r\_{f}$ so it should have a larger weight in $r\_{f}$ than M.

Since M is 1/3 in each of A, B and C, the weights in A, B and C are simply 1/3 of the weight in M:

$$x\_{A}=x\_{B}=x\_{C}=\frac{1}{3}×x\_{M}$$

$$ =\frac{1}{3}×0.2=\frac{3}{15}=0.0666667$$

To find this efficient portfolio’s standard deviation, we could use the 2-stock portfolio variance equation with $r\_{f}$ and M and the weights we just found, together with the fact that the covariance of $r\_{M}$ with $r\_{f}$is zero since $r\_{f}$ is a constant:

$$σ\_{P}^{2}=x\_{1}^{2} .σ\_{1}^{2}+x\_{2}^{2} .σ\_{2}^{2}+2.x\_{1}.x\_{2}.σ\_{1,2}$$

But another faster method is to use the CML equation instead:

 $r\_{P}=\left(\frac{μ\_{m}-r\_{f}}{σ\_{m}}\right)σ\_{P}+r\_{f}$

Where $r\_{P}$ is the return of our efficient portfolio of 0.1, and $σ\_{P}$ is the variable we are trying to find.

$$0.1=\left(\frac{0.3-0.05}{0.2}\right)σ\_{P}+0.05$$

$σ\_{P}=0.04$