The St. Petersburg Paradox, from 1713

A coin-flipping game is offered at a casino.

You repeatedly toss a coin until you flip tails and then the game ends. So if you flip heads you keep flipping.

The payoff at the end of the game is \$2 raised to the power of how many times you flipped the coin. So the payoff is $\$2^n$, where n is the number of coin flips, including the tail flip that ended the game.

If you flip tails on your first go, you get \$2 (=2^1).

If you flip tails on your second go, you get \$4 (=2^2).

If you flip tails on your third go, you get \$8 (=2^3).

If you flip tails on your fourth go, you get \$16 (=2^4), etc.

Question 1: What price would you pay to participate in the game? Assume that you can only play once.

Question 2: What is the expected payoff from participating in this game once?

Expected Payoff of St Petersburg Paradox Game			
Number of coin flips	Probability	Payoff	Probability times payoff
1	0.5	2	1
2	0.25	4	1
3	0.125	8	1
4	0.0625	16	1
5	0.03125	32	1
6	0.015625	64	1
…	…	…	…
n	(1/2)^n	2^n	1

The probability multiplied by the payoff is always \$1. Since it's possible to keep flipping heads forever (say using a computer to simulate flips), the total expected payoff is equal to \$1 + \$1 + \$1 and so on for the potentially infinite amount of heads flips that are possible before flipping tails. So the expected payoff is infinite.

$$\text{Expected Payoff} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \times 2^n = \sum_{n=1}^{\infty} 1 = \infty$$

This is the paradox. No one would pay an infinite amount to participate in this game once. Most people would only pay a few dollars.

Utility

Daniel Bernoulli, the brother of Nicolas Bernoulli who first posed the paradox in 1712, proposed a resolution to the paradox in 1738. He observed that:

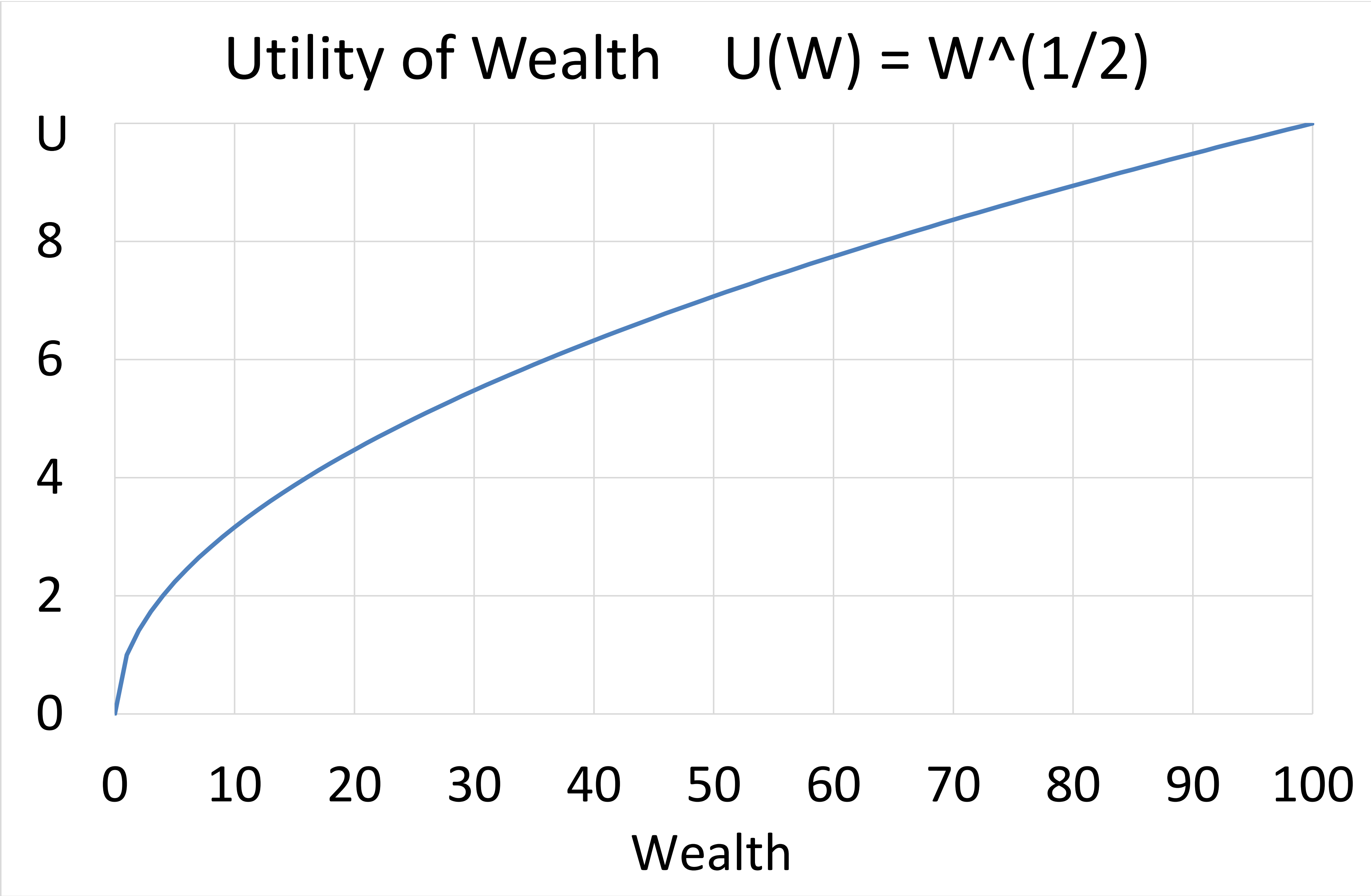
"The determination of the value of an item must not be based on the price, but rather on the utility it yields…. There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man though both gain the same amount."

Utility can be thought of as a measure of happiness.

Bernoulli argued that people judge utility as a function of money. The utility function can be used to measure the happiness from money, rather than the raw amount of money by itself.

Rational people should always prefer more money to less. Rational people are also thought to have diminishing marginal benefits from money. So your first $1,000 should make you more happy than the next $1,000.

For example, if a person's utility function was the square root of wealth, so $U(W) = W^{(1/2)}$, then their utility function would look like this.


Utility of Wealth U(W) = W^(1/2)
U
8
6
4
2
0
0
10
20
30
40
50
60
70
80
90
100
Wealth

This square root function offers a solution to the paradox.

Bernoulli used a log utility curve which looks quite similar to the square root utility function. It slopes up but at a diminishing rate. He suggested summing the probability of each event multiplied by the change in the utility of the player's wealth before and after the game.

Let the wealth before playing be W.

Let the cost of participating in a single game be C.

To find the maximum ticket price of the St Petersburg paradox game that a player with a natural logarithm utility curve would pay, find the change in the expected utility of wealth from playing the game once.

$$\text{ChangeInExpectedUtility} = \sum_{n=1}^{\infty} (Probability\ .\ UtilityChange)$$

$$= \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}\right)^{n} . (\ln(\text{WealthAfterGame}) - \ln(\text{WealthBeforeGame})) \right)$$

$$= \sum_{n=1}^{\infty} \left(\left(\frac{1}{2}\right)^{n} . (\ln(W + 2^{n} - C) - \ln(W)) \right)$$

Since the utility of the 'wealth before game' W is always the same no matter what the outcome, and the infinite sum of the probabilities that it's multiplied by is one, that term can be taken out of the sum.

$$\sum_{n=1}^{\infty} \left(\left(\frac{1}{2}\right)^{n} . \ln(W + 2^{n} - C) \right) - \ln(W)$$

This results in a finite expected utility from playing the St Petersburg paradox game. For example, a person with $1,000 in wealth would be willing to pay up to $10.95 to play the game once.

You can verify this using a spreadsheet program by calculating the sum of a long sequence of the probability-weighted utilities. Then use Goal Seek or Solver to set this sum equal to zero by changing the cost ‘C’. This cost C will be the maximum price that the player would pay to participate since zero utility would be gained.

Other Resolutions of the St Petersburg Paradox

Bernoulli's utility theory has been integral to the development of modern economic and financial theory. It's the dominant theory.

But keep in mind that this utility-theory solution to the St. Petersburg paradox is just one of several. Even today, more than 300 years later, new solutions are still being thought of and debated. Wikipedia has some interesting discussion about it: https://en.wikipedia.org/wiki/St._Petersburg_paradox#Recent_discussions