No-arbitrage Binomial Option Valuation

This approaches prices an option by making a portfolio that's short one option and long some fraction of a stock such that the portfolio has the same value regardless of whether the stock price rises or falls over the next period.

Therefore the portfolio is risk-free, so its value can be discounted to the present using the risk free-rate.

The current option price can now be calculated!

This will be based on the known stock price now, the risk-free rate, the fraction of a share required to make the portfolio risk-free, and the value of that portfolio next period.

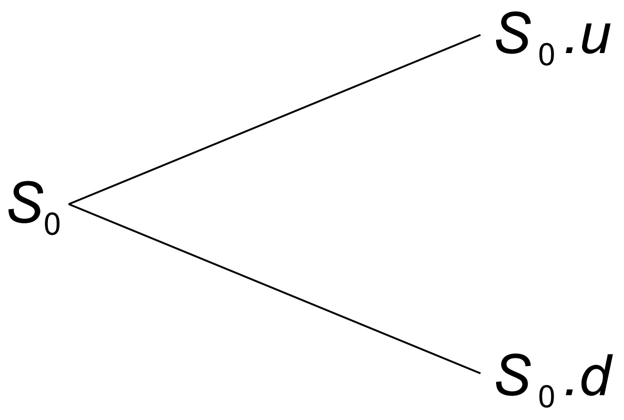
Let V_t be the value of a portfolio,

S_t is the value of the underlying stock,

f_t is the value of the option at time t,

u is one plus the proportional up move in the stock price (say 1+0.1), and

d is one minus the proportional down move in the stock price (say 1-0.1).



So after one period the stock price could be $S_{1u} = S_0.u$ if it went up or $S_{1d} = S_0.d$ if it went down.

No-arbitrage Valuation Steps

1. Make a portfolio V at $t = 0$ that's short 1 option and long Δ (delta) shares.
   $V_0 = -f_0 + \Delta. S_0$

2. Make 2 expressions for portfolio value at $t = 1$ for when the share price goes up (V_{1u}) and down (V_{1d}).
   $V_{1u} = -f_{1u} + \Delta. S_{1u}$
   $V_{1d} = -f_{1d} + \Delta. S_{1d}$
3. Set these expressions equal to each other and solve for Δ (Delta).
   $-f_{1u} + \Delta. S_{1u} = -f_{1d} + \Delta. S_{1d}$
   $\Delta = ?$

4. So Δ (Delta) is the fraction of a share, for every one option, that makes the whole portfolio worth the same amount whether the share price rises or falls.
   Since the portfolio is worth the same amount either way, there is no risk. Therefore we can discount the value of the portfolio by the risk free rate.
   By solving for f_0 we can find the current (t=0) price of the option!

   $$-f_0 + \Delta . S_0 = (-f_{1u} + \Delta . S_{1u})e^{-r\times 1} \quad \text{or}$$
   $$= (-f_{1d} + \Delta . S_{1d})e^{-r\times 1}$$

   $f_0 = ?$

Remember that the option price at maturity f_{1u} or f_{1d} can be easily calculated. In this example, the binomial tree is a single period of one year so t=1.

If, for example:

- The stock price rose and the option is a call, then:
  $\text{f}_{\text{1u call}} = \max(S_{1u} - K_1, 0)$
- The stock price fell and the option is a put, then:
  $\text{f}_{\text{1d put}} = \max(K_1 - S_{1d}, 0)$

Meaning of No-arbitrage

There is a no-arbitrage pricing relationship because if the risk-free portfolio of the short option and fraction of a share did not earn the risk-free rate, then there would be an arbitrage opportunity.

For example, if the portfolio had a lower expected return than the risk-free government bond, then you could sell the portfolio (long one option and short a fraction of a share) and use the money to buy government bonds.

This would make you a risk free return without investing any money yourself!

But as you did this trade, you would sell the portfolio price down (which means the option price would rise and the share price would fall) and bid the government bond price up until there was no arbitrage opportunity and everything was fairly priced.