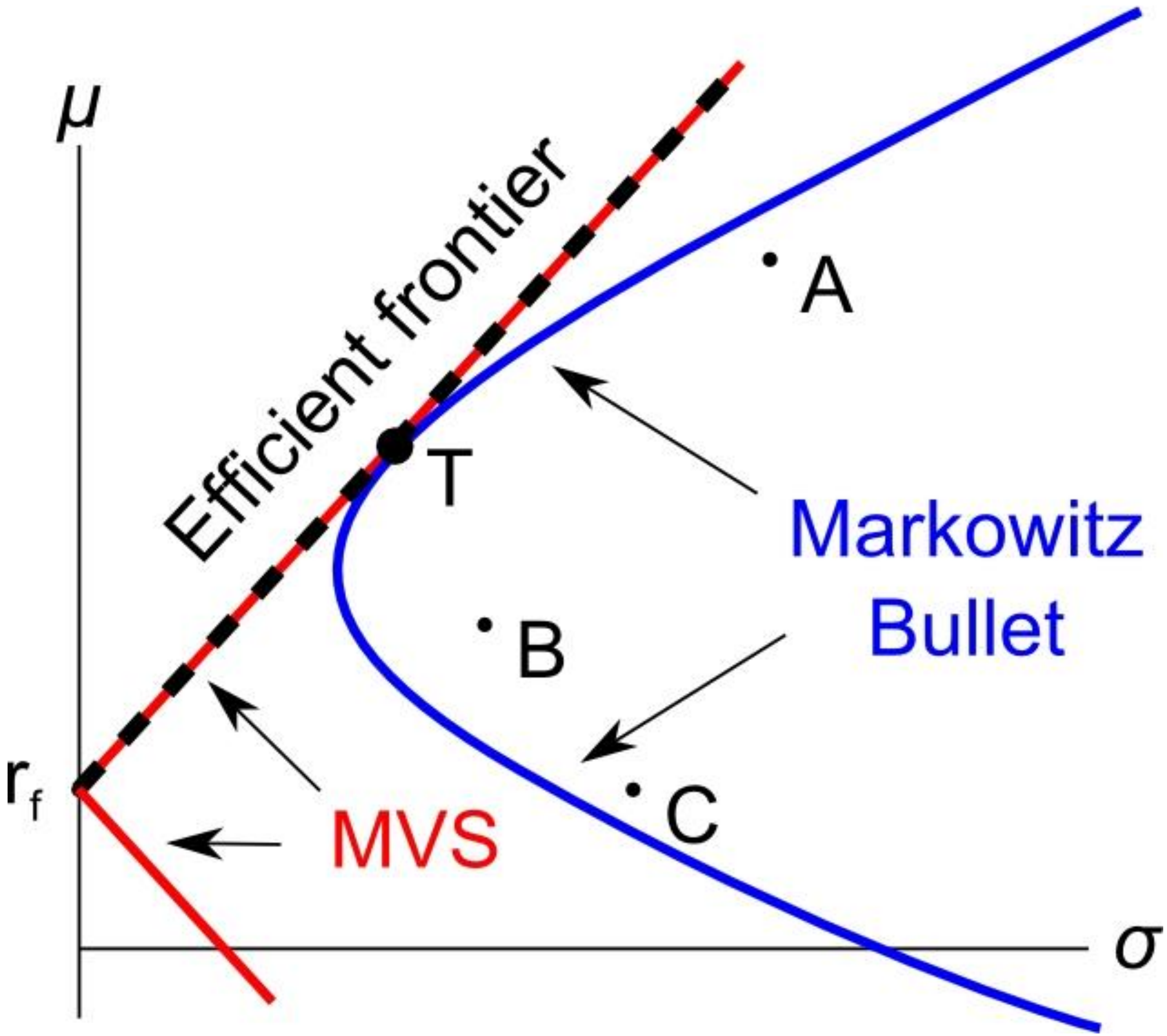
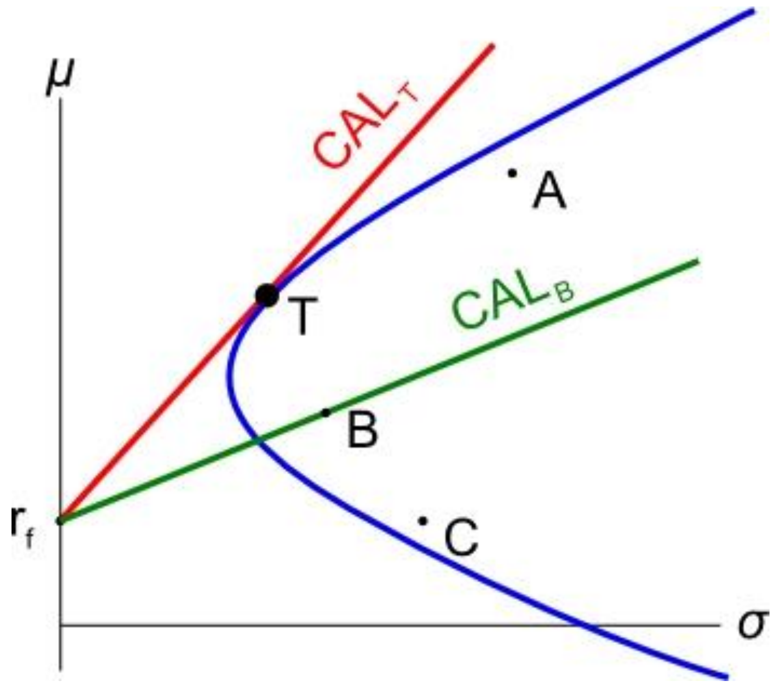
The Risk Free Rate (r_f) and the Minimum Variance Set of All Assets

- Risk-free securities have zero standard deviation of returns. Government bonds (also called Treasuries) are assumed to be risk free securities.
- The return of the risk-free security is referred to as r_f. It's also used to refer to the security itself.
- When r_f is included, the **new MVS becomes a line** from r_f through the tangency portfolio (T) on the Markowitz bullet.
- r_f is a constant, so it has zero variance, and zero covariance with other securities.
- $r_f = E(r_f) = \mu_{rf}$, since r_f is a constant.


μ
Efficient frontier
A
T
Markowitz
Bullet
B
C
r_f
MVS
σ

The Tangency Portfolio

- The tangency portfolio T is the only risky portfolio worth investing in. It is comprised of the stocks A, B and C.
- For any level of risk (standard deviation), the highest return possible can be achieved by investing in T and r_f.
- Lines from r_f through any portfolio are called Capital Allocation Lines.
- The CAL through T has the best risk-return trade off. It has the steepest gradient (rise/run).
- The gradient of the CAL is also called the Sharpe ratio.

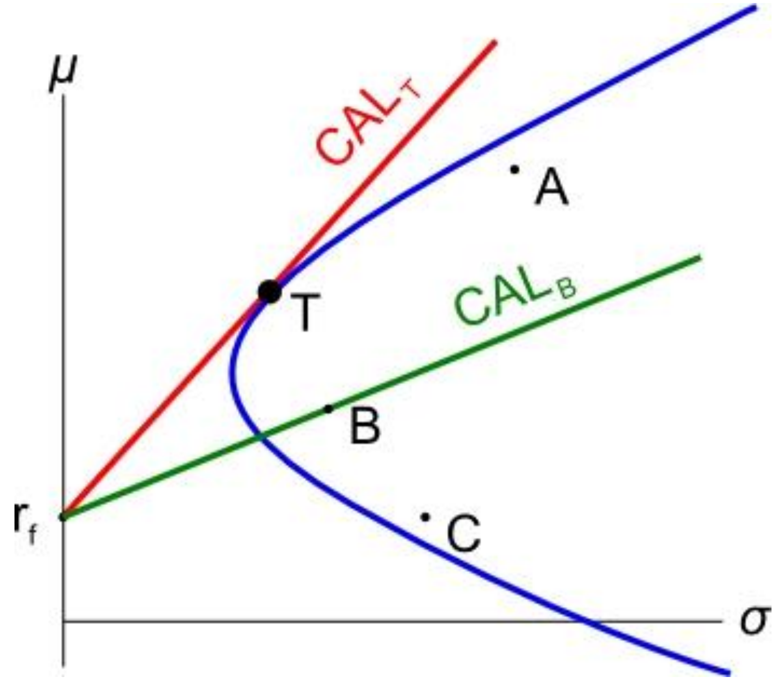


Sharpe Ratio

The **Sharpe ratio** (S) of a stock is the gradient of the line from r_f through the stock. It is the gradient of the stock's CAL.

$$S_i = \frac{\mu_i - r_f}{\sigma_i}$$

Where S_i is the Sharpe ratio of stock 'i', μ_i is its expected return and σ_i is its standard deviation.



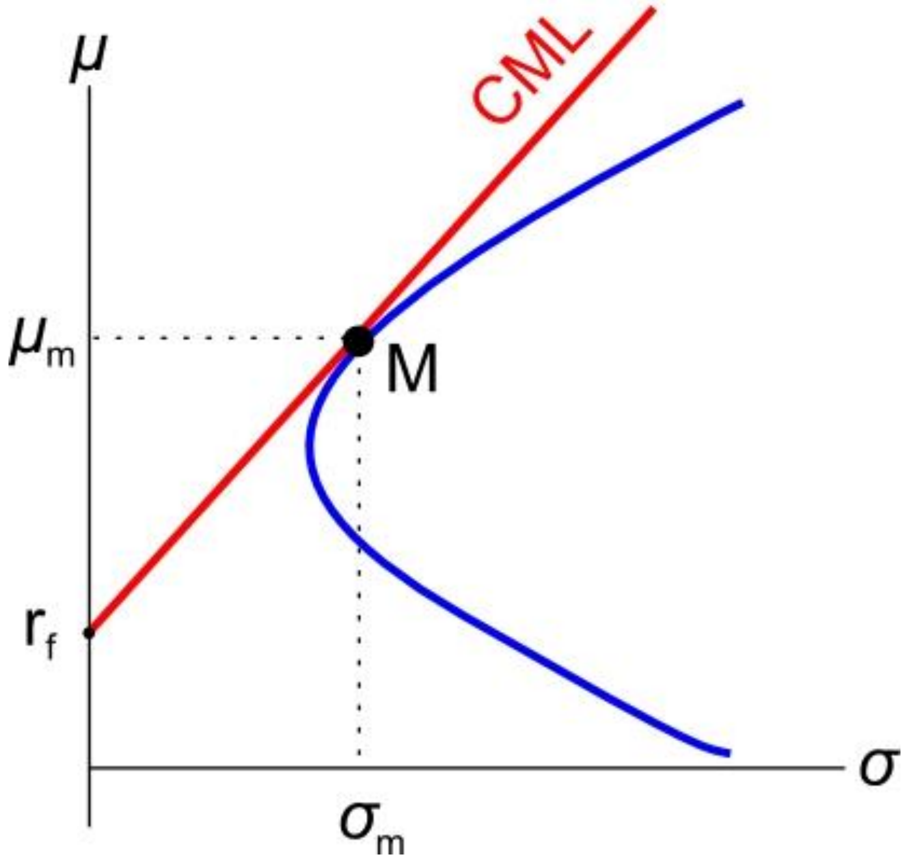
In the diagram, portfolio T's Sharpe ratio is greater than stock B's since the CAL through T is steeper. Therefore portfolio T is preferable to stock B.

The Market Portfolio

- The market portfolio M is the tangency portfolio of **all** risky assets.

- The line through M and r_f is called the Capital Market Line (CML).

- The CML has the steepest gradient, therefore the market portfolio has the highest Sharpe ratio.

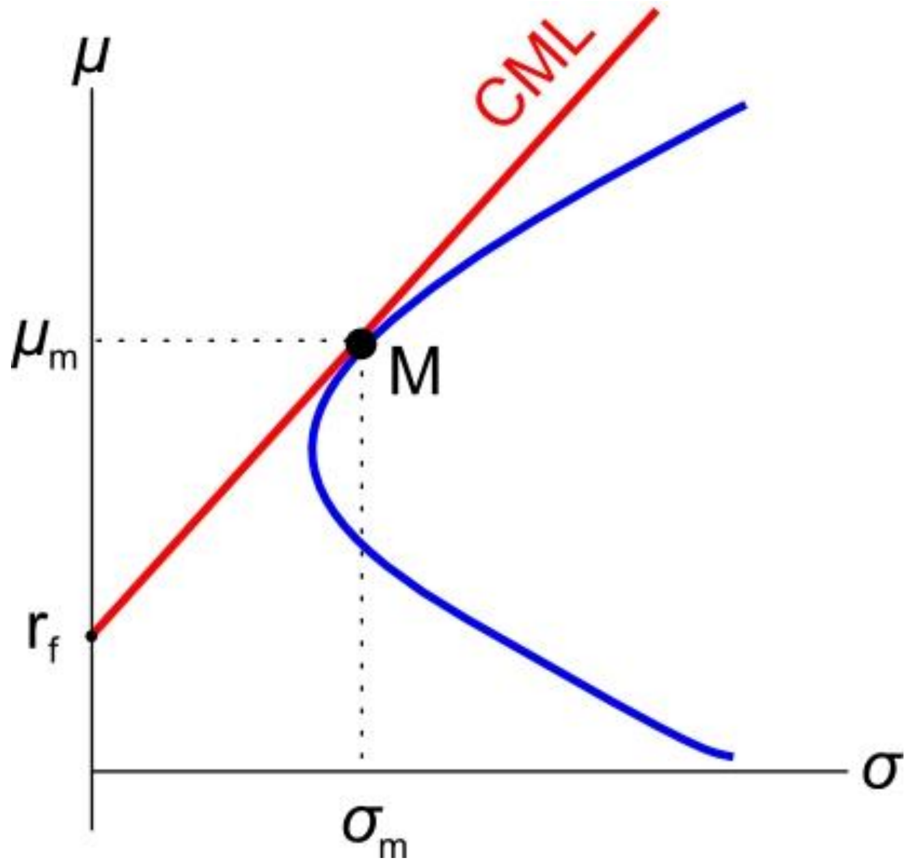


Equation of the Capital Market Line (CML)

$$\mu = r_f + \sigma\left(\frac{\mu_m - r_f}{\sigma_m}\right)$$

This is easy to see since the:

- y-axis is expected return μ
- x-axis is standard deviation σ
- y-intercept is r_f
- gradient $\frac{rise}{run}$ between r_f and M is $\frac{\mu_m - r_f}{\sigma_m}$



So in $y = mx + b$ form:

$$\mu = \left(\frac{\mu_m - r_f}{\sigma_m}\right)\sigma + r_f$$

Calculation Example

Question: Assume a 3-stock world consisting of A, B and C, as well as the risk free security. The market portfolio has been calculated to have weights 1/3 in each of A, B and C. The risk free rate $r_f = 0.05$, the market return is $r_m = 0.3$ and the market's standard deviation is $\sigma_m = 0.2$.

Find the weights in stocks A, B, C and r_f which makes an efficient portfolio (P) with a return of $r_{p\ target} = 0.1$. Also find this portfolio's standard deviation $\sigma_{p\ target}$.

Answer: An efficient portfolio has minimum variance (or st. dev.) for a given return. All portfolios on the CML are efficient.

Therefore we need only consider investing in the market portfolio (M) and the risk free rate (r_f).

To find the weights we need to invest in M and r_f, we will use the portfolio return equation, with a target portfolio return of 0.1:

$$r_P = x_1.r_1 + x_2.r_2 + \cdots + x_n.r_n$$

$$0.1 = x_M \times 0.3 + x_{rf} \times 0.05$$

Now we're stuck since we have 2 unknowns (x_M and x_{rf}) and only one equation so we can't find either of the weights. But there is another equation, the 'sum of the weights equals one':

$$x_1 + x_2 + \cdots + x_n = 1$$

$$x_M + x_{rf} = 1$$

$x_{rf} = 1 - x_M$

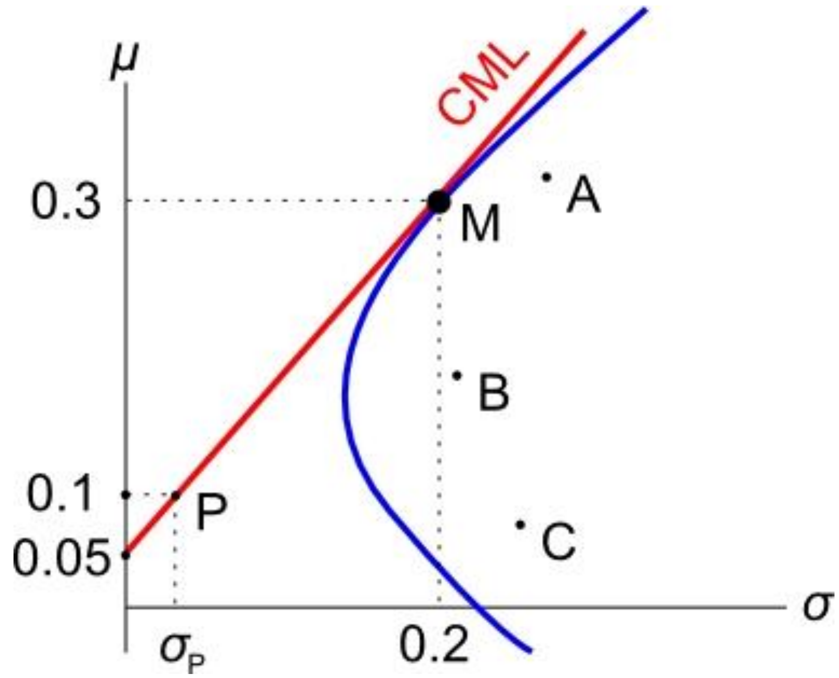
Substitute this into the portfolio return equation to get:

$0.1 = x_M \times 0.3 \; + (1 - x_M) \times 0.05$

$x_M = 0.2$

So, $\quad x_{rf} = 1 - 0.2 = 0.8$

This makes sense since the target return of 0.1 is closer to r_f so it should have a larger weight in r_f than M.



Since M is 1/3 in each of A, B and C, the weights in A, B and C are simply 1/3 of the weight in M:

$$x_A = x_B = x_C = \frac{1}{3} \times x_M$$

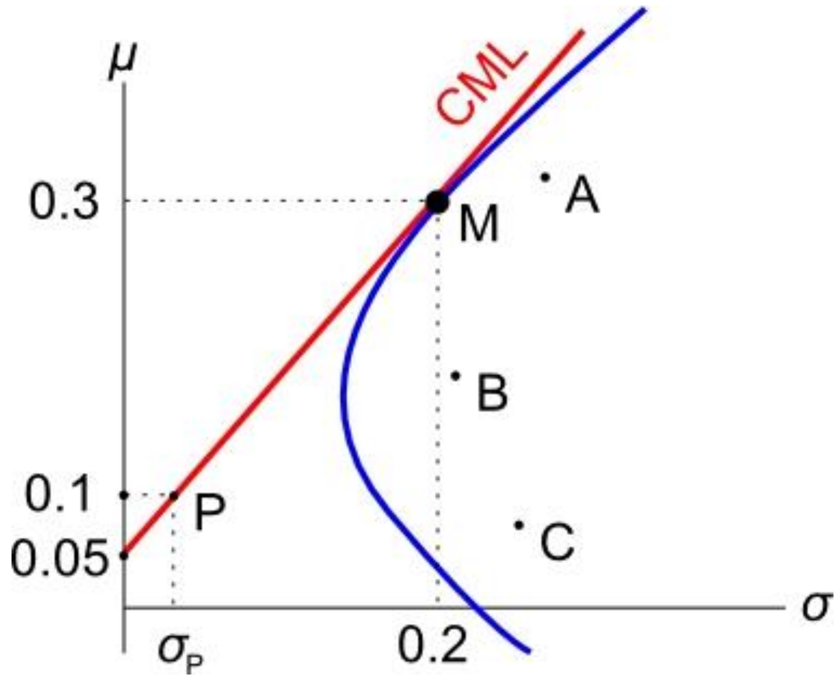
$$= \frac{1}{3} \times 0.2 = \frac{3}{15} = 0.0666667$$

To find this efficient portfolio's standard deviation, we could use the 2-stock portfolio variance equation with r_f and M and the weights we just found, together with the fact that the covariance of r_M with r_f is zero since r_f is a constant:

$${\sigma_P}^2 = x_1^2 \, . \, \sigma_1^2 + x_2^2 \, . \, \sigma_2^2 + 2. \, x_1 . \, x_2 . \, \sigma_{1,2}$$

But another faster method is to use the CML equation instead:

$$r_P = \left(\frac{\mu_m - r_f}{\sigma_m}\right)\sigma_P + r_f$$



Where r_P is the return of our efficient portfolio of 0.1, and σ_P is the variable we are trying to find.

$$0.1 = \left(\frac{0.3 - 0.05}{0.2}\right)\sigma_P + 0.05$$

$$\sigma_P = 0.04$$